Indian Statistical Institute, Bangalore
B. Math (Hons.) Second Year
Second Semester - Ordinary Differential Equations
Mid-Semester Exam

Maximum Marks: 30 **Date:** February 25, 2026
Duration: 2 hours

ANSWER ALL QUESTIONS

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1. Solve the following equation: [5]

$$(x^4 + y^2)dy - (4x^3y)dx = 0.$$

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by [5]

$$f(x,y) = \begin{cases} \frac{4x^3y}{x^4+y^2}, & \forall (x,y) \neq (0,0), \\ 0, & \text{for } (x,y) = (0,0). \end{cases}$$

What can you conclude about the existence/uniqueness of solution of $\dot{y} = f(x,y)$ in a neighbourhood of 0 satisfying $y(0) = 0$ using:
   (a) Peano's theorem.
   (b) Picard's theorem.

3. Find the general solution of the following differential equation:

$$y''(x) - y'(x) = e^x.$$

Justify that any solution is of the form you claim above.

OR [5]

Find the general solution of the following differential equation:

$$y''(x) + 10y'(x) - 10y(x) = e^x.$$

4. Let $p : \mathbb{R} \to \mathbb{R}$ be a continuous function. Consider the following differential equation:

$$y''(x) + p(x)y(x) = 0. \qquad (\star)$$

   (a) Let y_1 and y_2 be two independent solutions of ($\star$), prove that the Wronskian W_{y_1,y_2} is constant. [2]

   (b) Let y be a solution of ($\star$) in $\mathbb{R}$ satisfying

$$y(1/n) = 0, \ \forall n \in \mathbb{N}. \qquad (\star\star)$$

   Then what are the values of y and y' at 0? Use this information to conclude that there is no non-constant solution of ($\star$) which satisfies ($\star\star$). [3]

5. Solve the following system of differential equations for $Y = (Y_1, Y_2)$:

$$\dot{Y} = \begin{bmatrix} 1 & e^{-x} \\ 0 & 2 \end{bmatrix} Y,$$

satisfying $Y_1(0) = Y_2(0) = 1$. [5]

6. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $y : \mathbb{R} \to \mathbb{R}$ solve the differential equation $\dot{y} = f(y)$ on $\mathbb{R}$. Further assume that

$$\lim_{x\to\infty} y(x) = a \quad \& \quad y(0) = 0,$$

where $a \in \mathbb{R}$. Prove that $f(a) = 0$ and $\lim_{x\to\infty} y'(x) = 0$. [3.5]

(b) Prove/disprove: Let $h : (1, \infty) \to \mathbb{R}$ be a C^1 function that $\lim_{x\to\infty} h(x) = a$, for some $a \in \mathbb{R}$. Then $\lim_{x\to\infty} h'(x) = 0$. [1.5]

Good luck!!